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REAL OPERATOR ALGEBRAS*

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Abstract. This paper is a summary of my works on real operator algebras, which contains the following: Definitions of real C^*-algebras and real W^*-algebras, Gelfand-Naimark conjecture in real case, A proof of the structure theorem of finite dimensional real C^*-algebras in operator algebra method, Irreducible $*$ representations of real C^*-algebras, The classification of real Von Neumann algebras.

§1. Introduction

As well-known, the theory of (complex) operator algebras is very rich and important. So it is a natural and interesting problem: what's happen in real case?

A natural way to real case is as follows. Let A be a real $*$ algebra. Then $A_c = A \dot{+} iA$ is a complex $*$ algebra in a natural manner. Consider A_c and then go back to A. Moreover, for any $x \in A$ the spectrum $\sigma(x)$ of x is the spectrum of x as an element of A_c. In particular, $\overline{\sigma(x)} = \sigma(x)$.

In this paper, we study some fundamental results of real operator algebras.

A (complex) C^*-algebra B is a (complex) Banach $*$ algebra and $\|x^*x\| = \|x\|^2, \forall x \in B$. But the definition of a real C^*-algebra needs some additional condition. We give the definitions of real C^*-algebras and real W^*-algebras in §2.

Gelfand-Naimark conjecture ([1]) is very important for the theory of (complex) C^*-algebras, i.e., could the condition $\|x^*x\| = \|x\|^2 (\forall x \in B)$ be replaced by a weaker condition $\|x^*x\| = \|x^*\| \cdot \|x\| (\forall x \in B)$ for a C^*-algebra B? In S3 we discuss this conjecture in real case.

It is well-known that any divisible real Banach algebra is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (quaternion field). Its purely algebraic proof depends on the Wedderburn theorem.

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L.E.Dickson ([2]) gave a proof in Banach algebra method. Samely, the proof of the structure theorem of finite dimensional real C^*-algebras in [3] is purely algebraic and still depends the wedderburn theorem. In §4 we sketch a proof in operator algebra method.

For a topologically irreducible $*$ representation of a (complex) C^*-algebra the n-transitivity ([4]) holds for any n. Consequently, a topologically irreducible $*$ representation is also algebraically irreducible. But in real case the n-transitivity is not true for $n \geq 2$ generally. In §5 we point out that 1-transitivity still holds in real case. In particular, a topologically irreducible $*$ representation of a real C^*-algebra is still algebraically irreducible.

In §6, we discuss the Von Neumann-Murray classification of real Von Neumann algebras. For first classification the situation is similar to complex case. But in second classification some new situation appears.

§2. Definitions of real operator algebras

Definition 2.1. ([5]) A real Banach $*$ algebra is called a real C^*-algebra, if $A_c = A \dot{+} iA$ can be normed to become a (complex) C^*-algebra and keep the original norm on A.

Let A be a real C^*-algebra, and $\mathcal{S}(A)$ the real state space on A. For any $\varphi \in \mathcal{S}(A)$ we have GNS construction $\{H_\varphi, \pi_\varphi, \xi_\varphi\}$. Further, let

$$H = \sum_{\varphi \in \mathcal{S}(A)} \oplus H_\varphi, \qquad \pi = \sum_{\varphi \in \mathcal{S}(A)} \oplus \pi_\varphi.$$

Then A is isometrically $*$ isomorphic to $\pi(A)$, and $\pi(A)$ is a uniformly closed $*$ operator algebra (concrete real C^*-algebra) on the real Hilbert space H ([5]).

Similar to the definition of a (complex) C^*-algebra, we have the following.

Theorem 2.2. (L. Ingelstam [6]) Let A be a real Banach $*$ algebra, and $\|x^*x\| = \|x\|^2, \forall x \in A$. Then A is a real C^*-algebra, if and only if, A is hermitian (i.e. for any $h^* = h \in A, \sigma(h) \subset \mathbb{R}$).

Let H be a real Hilbert space, and M a $*$ subalgebra of $B(H)$ (all bounded linear operators on H). Then M is called a real Von Neumann (VN, simply) algebra, if $1 \in M$ and M is weakly closed. It is easy to see that the Von Neumann's double commutation theorem and the Kaplansky's density theorem still hold for real VN algebras.

Definition 2.3. ([5]) A real C^*-algebra M is called a real W^*-algebra, if $M_c = M \dot{+} iM$ can be normed to become a (complex) W^*-algebra and keep the original norm on M.

Through all σ-continuous real states and the GNS construction, we can see that a real W^*-algebra can be $\sigma - \sigma$ continuously $*$ isomorphic to a real VN algebra. Moreover, if A is a real C^*-algebra, then A^{**} is a real W^*-algebra ([5]).

Similar to the definition of a (complex) W^*-algebra, we have the following.

Theorem 2.4. ([5]) Let M be a real C^*-algebra. Then M is a real W^*-algebra, if and only if, there exists a real Banach space M_* such that $M = (M_*)^*$ and the maps

$$\cdot \rightarrow a \cdot \text{ and } \cdot \rightarrow \cdot a : M \rightarrow M$$

are $\sigma - \sigma$ continuous, $\forall a \in M$.

Remark. Up to now, we don't know that the condition of $\sigma - \sigma$ continuity of maps $\cdot \rightarrow a\cdot$ and $\cdot \rightarrow \cdot a$ in Theorem 2.4 can be omitted. But in complex case the $\sigma - \sigma$ continuity of these maps is satisfied automatically ([7]).

§3 Gelfand-Naimark conjecture in real case

Theorem 3.1. (Glimm-Kadison [8]) Let B be a unital (complex) C^*-algebra, and S= $\{b \in B \mid \|b\| \leq 1\}$. Then

$$Co\{e^{ih} \mid h = h^* \in B\}$$

is dense in S.

By this theorem, Glimm and Kadison solved the Gelfand-Naimark conjecture in unital case. Further, Vowden ([9]) solved this conjecture in general case, i.e., we have the following.

Theorem 3.2. Let B be a (complex) Banach $*$ algebra, and $\|x^*x\| = \|x^*\| \cdot \|x\|, \forall x \in B$. Then B is a (complex) C^*-algebra.

In unital case, the condition "$\forall x \in B$ " can be weakened further.

Theorem 3.3. (Glickfeld [10]) Let B be a unital (complex) Banach $*$ algebra.

1) If there exists a constant $C(\geq 1)$ such that

$$\|e^{ih}\| \leq C, \quad \forall h^* = h \in B,$$

then B is C^*-equivalent.

2) If the constant $C = 1$ in 1), then B is a (complex) C^*-algebra.

3) If $\|x^*x\| = \|x^*\| \cdot \|x\|$ for each normal $x \in B$, then B is a (complex) C^*-algebra.

Elliott introduced the concept of strictly positive element, and then he omitted the unital condition.

Theorem 3.4. ([11]) Let B be a (complex) Banach $*$ algebra, and $\|x^*x\| = \|x^*\| \cdot \|x\|$ for each normal $x \in B$. Then B is a (complex) C^*-algebra.

All above results are in complex case. In real case, we have the following .

Theorem 3.5. ([5]) Let A be a unital real C^*-algebra, and S= $\{a \in A \mid \|a\| \leq 1\}$. Then

$$Co\{\cos b \cdot e^a \mid b^* = b, a^* = -a \in A\}$$

is dense in S.

By this theorem we solved the Gelfand-Naimark conjecture in real case.

Theorem 3.6. ([5]) Let A be a real Banach $*$ algebra, and $\|x^*x\| = \|x^*\| \cdot \|x\|, \forall x \in A$. If A is hermitian, then A is a real C^*-algebra.

In unital case, we have further result.

Theorem 3.7. ([5]) Let A be a unital real Banach $*$ algebra.

1) If there exists a constant $C(\geq 1)$ such that

$$\|\cos b\| \leq C, \quad \|e^a\| \leq C, \quad \forall b^* = b, \quad a^* = -a \in A,$$

then A is real C^*-equivalent.

2) If the constant $C = 1$ in 1), then A is a real C^*-algebra.

3) If $\|x^*x\| = \|x^*\| \cdot \|x\|$ for each normal $x \in A$, and A is hermitian, then A is a real C^*-algebra.

Remark. Up to now, we don't know that if A is non-unital then the conclusion 3) of Theorem 3.7 is still true.

§4 Finite dimensional real C^*-algebras

Let M be a real W^*-algebra, $U(M)$ the subset of all unitary elements of M, and $[U(M)]$ the (real) linear span of $U(M)$.

For any skew self-adjoint element $k \in M$ (i.e., $k^* = -k$), it is easy to see that $k \in [U(M)]$. For $h = h^* \in M$, let N be the real W^*-subalgebra of M generated by h and 1 (the identity of M). Then we can prove that

$$N \cong L_r^\infty(\Gamma, \nu).$$

Thus, $[U(N)]$ is σ-dense in N. From above discussion, we have the following.

Lemma 4.1. ([12]) Let M be a real W^*-algebra. Then $[U(M)]$ is σ-dense in M.

From this Lemma, the theorem of projection comparison ([4]) still holds in real W^*-algebras.

Now let A be a finite dimensional real C^*-algebra, and Z the center of A. Then

$$Z \cong C(\Omega, -), \quad {}^\sharp\Omega < \infty.$$

Hence, we can write that

$$(\Omega, -) = \{t_j, s_k, \bar{s}_k \mid 1 \le j \le n, 1 \le k \le m\},$$

where $\bar{t}_j = t_j, s_k \ne \bar{s}_k, \forall j, k$. Further,

$$A = \bigoplus_{j=1}^{n} A_j^{(1)} \oplus \oplus_{k=1}^{m} A_k^{(2)},$$

and $Z(A_j^{(1)}) \cong \mathbb{R}, Z(A_k^{(2)}) \cong \mathbb{C}, \forall j, k$.

Now we may assume that $Z \cong \mathbb{R}$ or $\mathbb{C}$.

1) $Z \cong \mathbb{R}$.

In this case, A is a finite dimensional real factor. Then we can take an orthogonal family $\{e_j \mid 1 \le j \le n\}$ of minimal projections of A such that $\sum_{j=1}^{n} e_j = 1$. By the theorem of projection comparison, $e_j \sim e_k, \forall j, k$. Thus we have that

$$A \cong M_n(\mathbb{R}) \overline{\otimes} pAp,$$

where $p = e_1$. It is easy to see that pAp is divisible. Therefore, $pAp \cong \mathbb{R}$ or $\mathbb{H}$.

2) $Z \cong \mathbb{C}$.

In this case, there exists $x \in Z$ such that

$$x^* = -x, \quad x^2 = -1$$

and $Z = \{\lambda + \mu x \mid \lambda, \mu \in \mathbb{R}\}$. Consider the (complex) C^*-algebra $A_c = A\dot{+}iA$ and its elements

$$z_1 = \frac{1}{2}(1\dot{+}ix), \quad z_2 = \frac{1}{2}(1\dot{+}i(-x)).$$

It is easy to see that

$$A_c = A_c z_1 \oplus A_c z_2, \quad \text{and } A_c z_j \cong M_{n_j}(\mathbb{C}),$$

$j = 1, 2$. Further, we can prove that

$$A \cong A_c z_1 \cong A_c z_2$$

as real C^*-algebras. Consequently, $n_1 = n_2 = n$, and $A \cong M_n(\mathbb{C})$.

Therefore, we proved the following structure theorem of finite dimensional real C^*-algebras in operator algebra method.

Theorem 4.2. ([3]) Let A be a finite dimensional real C^*-algebra. Then

$$A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k),$$

where $D_i = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}, 1 \le i \le k$.

§5 Irreducible $*$ representations of real C^*-algebras

Let B be a (complex) C^*-algebra, and $\{\pi, H\}$ a topologically irreducible $*$ representation of B. Then we have the following transitivity property ([4]): if $\xi_1, \cdots, \xi_n; \eta_1, \cdots, \eta_n \in H$ and $\{\xi_1, \cdots, \xi_n\}$ is linearly independent, then there exists $b \in B$ such that $\pi(b)\xi_i = \eta_i, 1 \le i \le n$. Consequently, $\{\pi, H\}$ is also algebraically irreducible.

However, the above transitivity property (for any n) is not true for real C^*-algebras generally. For example, consider the following real C^*-algebra A on real Hilbert space $\mathbb{R}^2$:

$$A = \{\lambda E + \mu U \mid \lambda, \mu \in \mathbb{R}\},$$

where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Clearly, A is irreducible on $\mathbb{R}^2$, but there are not $\lambda, \mu \in \mathbb{R}$ such that

$$(\lambda E + \mu U) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\lambda E + \mu U) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq 0.$$

Moreover, $A'' = A \neq M_2(\mathbb{R})$. And there is just one real state ρ on $A : \rho(\lambda E + \mu U) = \lambda, \forall \lambda, \mu \in \mathbb{R}$. Of course, ρ is pure. The null space N and left kernel I of ρ are $\{\mu U \mid \mu \in \mathbb{R}\}$ and $\{0\}$ respectively, and $N \neq I + I^*$. These are also different from the complex case.

In this section, we point out that 1-transitivity still holds in real case. In particular, a topologically irreducible $*$ representation is still algebraically irreducible for a real C^*-algebra.

Let A be a real C^*-algebra, and $A_c = A \dot{+} iA$. If ρ is a real state on A, then ρ_c is a state on the (complex) C^*-algebra A_c, where

$$\rho_c(a + ib) = \rho(a) + i\rho(b), \forall a, b \in A.$$

For any state φ on A_c, define $\overline{\varphi}$:

$$\overline{\varphi}(a + ib) = \overline{\varphi(a)} + i\overline{\varphi(b)}, \forall a, b \in A.$$

Clearly, $\overline{\varphi}$ is also a state on A_c, and $\overline{\overline{\varphi}} = \varphi$. Moreover, if φ is pure, then $\overline{\varphi}$ is also pure.

Proposition 5.1. ([13]) Let A be a real C^*-algebra, ρ a real state on A, and $\{\pi, H\}$ the $*$ representation of A generated by ρ.

1) If ρ is pure, then there exists a pure state φ on A_c such that

$$\rho_c = \frac{1}{2}(\varphi + \overline{\varphi}).$$

2) ρ is pure, if and only if, $\{\pi, H\}$ is topologically irreducible for A. In this case, $H = A/I$, where I is the left kernel of ρ.

It suffices to prove that $H = A/I$ if ρ is pure. And this proof can be gotten to follow from Halperin ([14]) essentially.

Proposition 5.2. ([13]) Let ρ be a pure real state on a real C^*-algebra A, and I, I_c the left kernels of ρ, ρ_c respectively. Let $\rho_c = \frac{1}{2}(\varphi + \overline{\varphi})$, where φ is a pure state on A_c, and $I_\varphi, I_{\overline{\varphi}}$ the left kernels of $\varphi, \overline{\varphi}$ respectively. Then

1) I is a regular closed left ideal of A;

2) $I_c = I_\varphi \cap I_{\overline{\varphi}}$;

3) I is a maximal left ideal of A.

The proofs of 1) and 2) are easy. Now on $H = A/I$, introduce two norms:

$$\|a + I\|_1 = \rho(a^*a)^{1/2}, \quad \|a + I\|_2 = \mathrm{dist}(a, I),$$

$\forall a \in A$. We can prove that they are equivalent. Further, if L s a maximal left ideal of A such that $I \subset L$, then L/I is not dense in H using $\|\cdot\|_1 \sim \|\cdot\|_2$. Therefore, $L = I$ and the proof is completed.

Remark. In complex case, $\|\cdot\|_1 = \|\cdot\|_2$ (Takesaki [15]).

Now we can prove the following.

Theorem 5.3. ([13]) Let A be a real C^*-algebra. Then there is a bijection between the collection of all pure real states on A and the collection of all regular maximal left ideals of A. Moreover, any closed left ideal L of A is the intersection of all regular maximal left ideals of A containing L.

Theorem 5.4. ([13]) Let A be a real C^*-algebra, and $\{\pi, H\}$ a topologically irreducible $*$ representation of A. Then for any $\xi, \eta \in H$ and $\xi \neq 0$ there is $a \in A$ such that

$$\pi(a)\xi = \eta.$$

Consequently, $\{\pi, H\}$ is also algebraically irreducible.

§6 The classification of real Von Neumann algebras

Let us consider the Von Neumann-Murray first classification of real VN algebras.

Let M be a real VN algebra. Then it is easy to see that we have the unique decomposition:

$$M = M_1 \oplus M_2 \oplus M_3,$$

where M_1, M_2, M_3 are finite, semi-finite and properly infinite, purely infinite real VN algebras respectively, and the concepts of finiteness, infiniteness of real VN algebras are the same as the complex case.

Now let M be a finite real VN algebra on a real Hilbert space H, Z the center of M, and M_h, Z_h the self-adjoint parts of M, Z respectively. Then we have a (real) linear map $T : M_h \to Z_h$ such that

$$\{T(a)\} = \overline{Co\{u^*au \mid u \in U(M)\}} \cap Z,$$

$\forall a \in M_h$, and

$$T(M_+) \subset Z_+, \quad T(z) = z, \quad T(a) = T(u^*au),$$

$\forall z \in Z_h, a \in M_h, u \in U(M)$, where M_+, Z_+ are the positive parts of M, Z respectively. Further, we can easily prove that M_c is also finite, where $M_c = M \dot{+} iM$ is a (complex) VN algebra on the (complex) Hilbert space $H_c = H \dot{+} iH$. Let

$$T_c : M_c \to Z_c$$

be the central valued trace of M_c, where $Z_c = Z \dot{+} iZ$ is the center of M_c. Then

$$T_c(\overline{x}) = \overline{T_c(x)}, \quad \forall x \in M_c,$$

where $\overline{x} = a - ib$ if $x = a + ib$ and $a, b \in M$. Consequently, $T_c(M) \subset Z, T_c|M_h = T$, and $T_c(M_k) \subset Z_k$, where M_k, Z_k are the skew self-adjoint parts of M, Z respectively. Therefore, we can define the central valued trace T from M onto Z as $T = T_c|M$.

Now let M be a semi-finite real VN algebra. If φ is a trace on M_+, then we can prove that there exists unique trace ψ on M_{c+} such that

$$\psi|M_+ = \varphi, \quad \psi(\overline{x}) = \psi(x), \quad \forall x \in M_{c+}.$$

Moreover, the definition ideal of ψ is $\mathcal{M}_c = \mathcal{M} \dot{+} i\mathcal{M}$, where $\mathcal{M}$ is the definition ideal of φ, and

$$\psi(a+ib) = \begin{cases} +\infty, & \text{if } (a+ib) \in M_c \backslash \mathcal{M}_{c+}, \\ \varphi(a), & \text{if } (a+ib) \in \mathcal{M}_{c+}, \end{cases}$$

where $a, b \in M$ Furthermore, φ is semi-finite, normal, or faithful, if and only if, so is ψ.

From the above discussion, we have the following.

Theorem 6.1. ([16]) A real VN algebra M is finite, properly infinite, semi-finite, or purely infinite, if and only if, the (complex) VN algebra $M_c = M \dot{+} iM$ is finite, properly infinite, semi-finite, or purely infinite.

Now we consider the Von Neumann-Murray second classification of real VN algebras. New situation appears, and it is different from the complex case.

Definition 6.2. ([17]) Let M be a real VN algebra, and $P(M)$ the subset of all projections of M.

$p \in P(M)$ is said to be abelian, if pMp is abelian;

$p \in P(M)$ is said to be semi-abelian, if $pM_h p$ is abelian.

M is said to be discrete, if for any non-zero central projection z of M there is a non-zero abelian projection p of M such that $p \leq z$;

M is said to be semi-discrete, if for any non-zero central projection z of M there is a non-zero semi-abelian projection p of M such that $p \leq z$;

M is said to be semi-continuous, if there is no any non-zero abelian projection in M;

M is said to be continuous, if there is no any non-zero semi-abelian projection in M.

Remark. In complex case, a semi-abelian projection must be abelian. But in real case, they can be different. For example, 1 is a non-abelian but semi-abelian projection of the real VN algebra $\mathbb{H}(\mathbb{H}_h = \mathbb{R}1)$.

Theorem 6.3. ([17]) Let M be a real VN algebra. Then we have the unique decomposition:

$$\begin{aligned} M = &\; M_1 \oplus \tilde{M}_2 \oplus M_3 = \tilde{M}_1 \oplus M_2 \oplus M_3 \\ = &\; M_1 \oplus M_{1,2} \oplus M_2 \oplus M_3, \end{aligned}$$

where M_1 is discrete (type I), $\tilde{M}_2$ is semi-finite and semi-continuous, $\tilde{M}_1$ is semi-discrete, M_2 is semi-finite and continuous (type II), $M_{1,2}$ is semi-discrete and semi-continuous, $\tilde{M}_1 = M_1 \oplus M_{1,2}, \tilde{M}_2 = M_{1,2} \oplus M_2$, and M_3 is purely infinite (type III).

Remark. $M_{1,2}$ is existential, for example, $L_r^\infty(\Gamma, \nu)\overline{\otimes}\mathbb{H}$, and it is necessary to study it further. Moreover, except type I, II, III real factors we also have semi-discrete and semi-continuous real factors, and it must be

$$B(H_n)\overline{\otimes}\mathbb{H},$$

where H_n is a n-dimensional real Hilbert space, and n is finite or infinite.

Proposition 6.3. ([17]) Let M be a real VN algebra, and $M_c = M\dot{+}iM$. Then we have the following relations.

1) M discrete $\Longleftrightarrow M'$ discrete

$\Longrightarrow M_c$ discrete

$\Longrightarrow M$ and M' semi-discrete;

2) M semi-continuous $\Longleftrightarrow M'$ semi-continuous;

3) M continuous $\Longrightarrow M_c$ continuous

$\Longrightarrow M$ semi-continuous.

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